Schwarz domain decomposition and domain truncation for exterior time-harmonic problems with variable coefficients and convective flows

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June 23, 2025

29th International Conference on Domain Decomposition Methods

Joint work : X. Antoine (U. Lorraine), C. Geuzaine (U. Liège), H. Bériot (Siemens)

Outline

1. Time-harmonic problems with convection
2. Domain truncation for exterior problems
3. Schwarz domain decomposition for convected propagation
4. Conclusion

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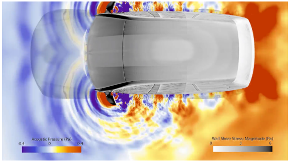
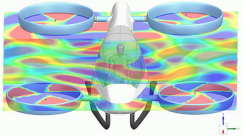
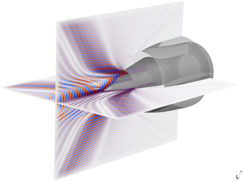
Aeroacoustics in the transport industry







Aeroacoustics studies the generation and propagation of sound in **moving fluids**

A simple model : sound propagation in a mean flow $\rightarrow$ convected propagation

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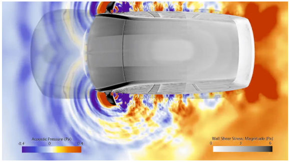

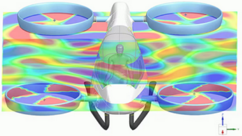

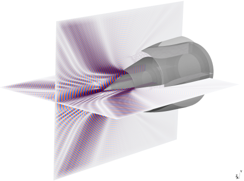

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A simple model : sound propagation in a mean flow $\rightarrow$ convected propagation

Time-harmonic convected wave operator *[Pierce 1990, Spieser, Bailly 2020]*

$$\mathcal{P} = -\rho_0 \mathrm{D}_{\boldsymbol{v}_0} \left(\frac{1}{{\rho_0}^2 {c_0}^2} \mathrm{D}_{\boldsymbol{v}_0} \right) + \nabla \cdot \left(\frac{1}{\rho_0} \nabla \right), \quad \mathrm{D}_{\boldsymbol{v}_0} = \mathrm{i}\omega + \boldsymbol{v}_0 \cdot \nabla$$

Mathematical properties

- Helmholtz-type operator with varying $c_0(\boldsymbol{x})$, $\rho_0(\boldsymbol{x})$ and mean flow $\boldsymbol{v}_0(\boldsymbol{x})$
- $\mathcal{P}$ is scalar and self-adjoint,
- If $c_0(\boldsymbol{x}) = \rho_0(\boldsymbol{x}) = 1 \Rightarrow$ convected Helmholtz, $\boldsymbol{v}_0(\boldsymbol{x}) = 0 \Rightarrow$ Helmholtz

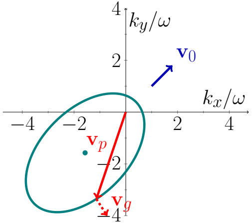
The physics of convected wave propagation

Plane-wave dispersion analysis : $u(\boldsymbol{x}) = e^{-\imath \boldsymbol{k}\cdot\boldsymbol{x}}$, $\boldsymbol{k} = (k_x, k_y)^T$

Convected Helmholtz operator $\mathcal{P} = -(\imath\omega + \boldsymbol{v}_0 \cdot \nabla)^2 + \Delta$, s.t. $\mathcal{P}u = 0$

Convected Helmholtz $\mathcal{P}$

$$(\omega - \boldsymbol{v}_0 \cdot \boldsymbol{k})^2 - |\boldsymbol{k}|^2 = 0$$



$\boldsymbol{v}_0 = 0.8 \times (\cos(\pi/4), \sin(\pi/4))^T$

- Group velocity is driven by the flow : $\boldsymbol{v}_g = \boldsymbol{v}_0 + c_0\boldsymbol{k}/|\boldsymbol{k}|$

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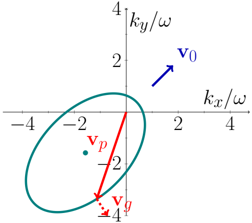
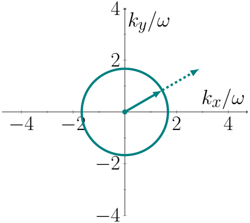
Helmholtz $\hat{\mathcal{H}}$

$|\boldsymbol{k}|^2 - \hat{\omega}^2 = 0$



$\longrightarrow$

Lorentz transform



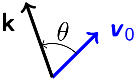
$\boldsymbol{v}_0 = 0.8 \times (\cos(\pi/4), \sin(\pi/4))^T$

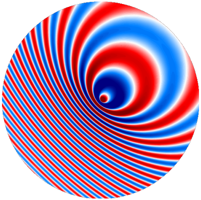
$\hat{\omega} = \omega / \sqrt{1 - |\boldsymbol{v}_0|^2 / c_0^2}$

- Group velocity is driven by the flow : $\boldsymbol{v}_g = \boldsymbol{v}_0 + c_0 \boldsymbol{k}/|\boldsymbol{k}|$
- The *Lorentz transform* maps $\mathcal{P}$ to $\hat{\mathcal{H}}$ [*Taylor 1978, Hu et al. 19, Barucq et al. 22*]

Numerical challenges for convected propagation

The mean flow impacts wave propagation
$\Rightarrow$ we must adapt numerical methods





Green kernel, $M = |\boldsymbol{v}_0|/c_0 = 0.8$

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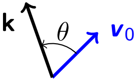

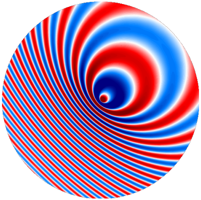

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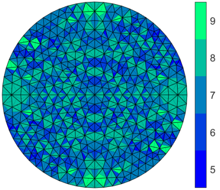

A priori p-FEM order adaptation [*Bériot, Gabard 19*]

Numerical challenges

- Discretization: dispersion error is affected [*Bériot et al. 12, Ainsworth 2004*]

$$E_d = \frac{1-M\cos(\theta)}{2}\left[\frac{p!}{(2p)!}\right]^2 \frac{1}{2p+1}\frac{(\omega h)^{2p+1}}{(1+M\cos(\theta))^{2p+1}} + \mathcal{O}(\omega h)^{2p+3},\ \omega h \to 0$$

$\rightarrow$ high-order is advocated: choose $d_\lambda^* = \frac{2\pi p}{\omega h}(1-M) \approx 6$

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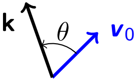

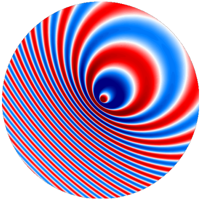

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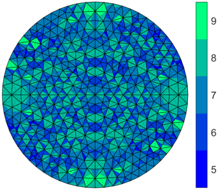

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  $\rightarrow$ high-order is advocated: choose $d_\lambda^* = \frac{2\pi p}{\omega h}(1-M) \approx 6$
- **Domain truncation**: phase and group velocity have different directions

$\rightarrow$ high-frequency solver : use **domain truncation** to build preconditioner for iterative methods

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Microlocal factorization

[*Engquist, Majda 1977*] construction : cancel bi-characteristics on the boundary

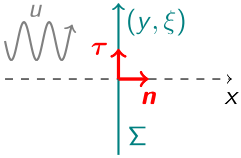
1. Split convected wave operator $\mathcal{P}$ into bi-characteristics [*Nirenberg 1973*]

$$\mathcal{P} = \left(\partial_x + \imath\Lambda^-\right)\left(\partial_x + \imath\Lambda^+\right) + \mathcal{R}$$

   The operators $\Lambda^\pm$ map the Dirichlet-to-Neumann data on Σ

2. canceling one of the factors on Σ gives a non-reflecting boundary condition

Half-space setting



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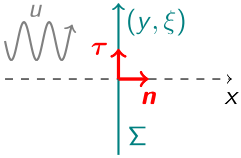


$\rightarrow$ Identify with the PDE operator to obtain a Ricatti equation for Λ^+

$$\left(1 - {M_x}^2\right)\left[\left(\Lambda^+\right)^2 + \imath\mathrm{Op}\left\{\partial_x\lambda^+\right\}\right] + \imath(\mathcal{A}_1 + \mathcal{A}_0)\Lambda^+ = \mathcal{B}_2 + \mathcal{B}_1,\ M_x = v_x/c_0$$

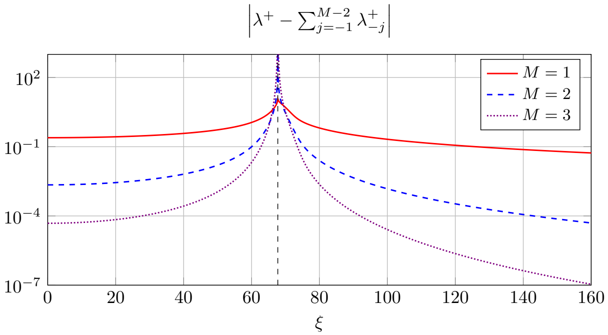
$\Lambda^+ = \mathrm{Op}(\lambda^+)$ is a ψDO associated to the symbol λ^+

Use a “high-frequency” asymptotic expansion $\lambda^+ \sim \lambda_1^+ + \lambda_0^+ + \cdots$, and compute each λ_{-j}^+ with homogeneity degree $(\omega, \boldsymbol{\xi})^{-j}$ [*Hörmander 2007*]

DtN symbol expansion for a Helmholtz problem

Symbol calculation with $\boldsymbol{v}_0 = 0$, $\rho_0 = 1$, $c_0^{-2}(x) = ax + b$, $\omega = 30$

Analytic symbol available $\lambda^+ = -ie^{-\frac{2i\pi}{3}} \left(a\omega^2\right)^{1/3} \frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}, \quad z = e^{-\frac{2i\pi}{3}} \frac{\xi^2 - \omega^2(ax+b)}{\left(a\omega^2\right)^{2/3}}$



- $\lambda_1^+ = \sqrt{\omega^2 c_0^{-2}(x) - \xi^2}$ is the "usual" square-root
- λ_0^+ depends on $\partial_x(c_0^{-2})$, matches the Airy function asymptotic expansion
- λ_{-1}^+ depends on $\partial_x^2(c_0^{-2})$ and $[\partial_x(c_0^{-2})]^2$, etc.

Principal symbol for convected propagation

Principal symbol for the half-space problem

$$\lambda_1^+ = \frac{1}{1-{M_x}^2}\left[-M_x(k_0 - \boldsymbol{M_\tau}\cdot\boldsymbol{\xi}) + \sqrt{(k_0 - \boldsymbol{M_\tau}\cdot\boldsymbol{\xi})^2 - (1-{M_x}^2)|\boldsymbol{\xi}|^2}\right]$$

with $k_0 = \omega/c_0$, $\boldsymbol{M_\tau} = \boldsymbol{v}_0\cdot\boldsymbol{\tau}$. λ_1^+ depends on local flow properties

- λ_1^+ matches the dispersion relation of a plane wave in a uniform flow
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We do the DtN approximation $\Lambda^+ \approx \mathrm{Op}\left(\lambda_1^+\right)$, and neglect λ_0^+, λ_{-1}^+, etc.
$\rightarrow$ flow variations and curvature effects are in the next symbols

A choice of operator representation

$$\mathrm{Op}\left(\lambda_1^+\right) = \frac{1}{1-M_x^2}\left[-M_x\left(k_0 - \imath\boldsymbol{M_\tau}\cdot\nabla_\Gamma\right) + \sqrt{(k_0 - \imath\boldsymbol{M_\tau}\cdot\nabla_\Gamma)^2 + (1-M_x^2)\Delta_\Gamma}\right]$$

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How to approximate $Op(\lambda_1^+)$ by a local operator ?

Operator approximations

Op $\left(\lambda_1^+\right)$ has a non-local term $f(Z) = \sqrt{1+Z}$, with $Z \to 0$ at high frequency
We can use a Taylor/Padé expansion $\Rightarrow$ sparse discretization

$$\left.\begin{array}{l} \bullet \text{ propagating modes live in } I = (-1, +\infty) \\ \bullet \text{ evanescent modes live in } I = (-\infty, -1) \end{array}\right\} \text{branch-cut problem}$$

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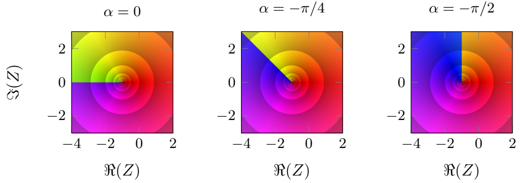

$f(Z)$ with different branch-cut rotations

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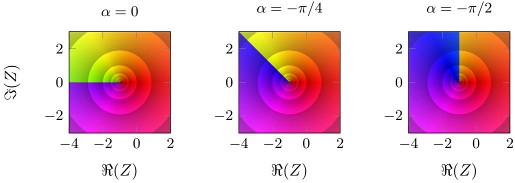

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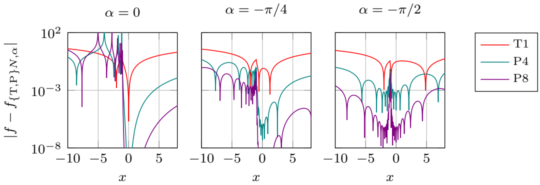


approximations of $f(Z)$ along the real line

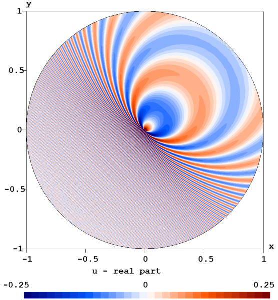
- computational cost increase with Padé order
- uniform rational approximation ?
  $\to$ Zolotarev solution
  [*Druskin et al. 2016*]

Exterior domain truncation - ABC

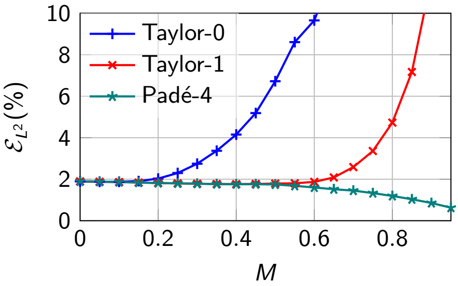
Our boundary condition reads $\partial_n u = -\imath \mathrm{Op}(\lambda_1^+) u$

$\rightarrow$ implementation in a Galerkin formulation with p-FEM

Example: absorbing boundary condition for convex boundary shape



(a) $\Re(u)$, M$= 0.95$, $\omega = 6\pi$



(b) Relative domain L^2-errors (in %)

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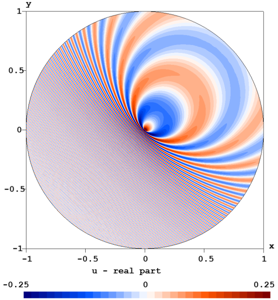

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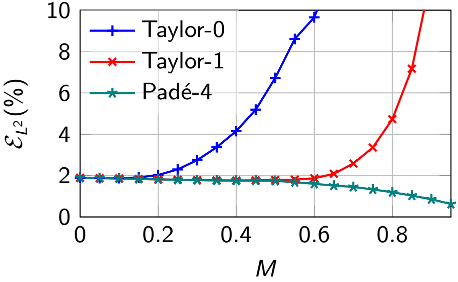

(b) Relative domain L^2-errors (in %)

- Microlocal construction allows to design high-order ABCs
- including the correction term λ_0^+ requires technical effort

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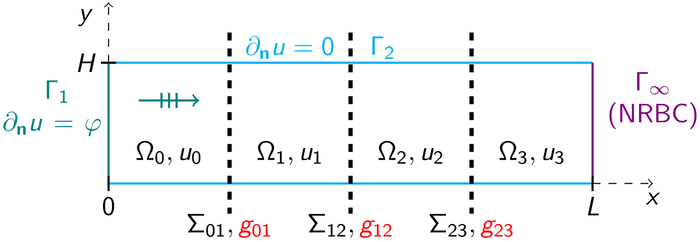
Non-overlapping Schwarz method

Semi-open waveguide configuration, propagation along the x-direction



The source φ is a superposition of 30 modes

Non-overlapping Schwarz method

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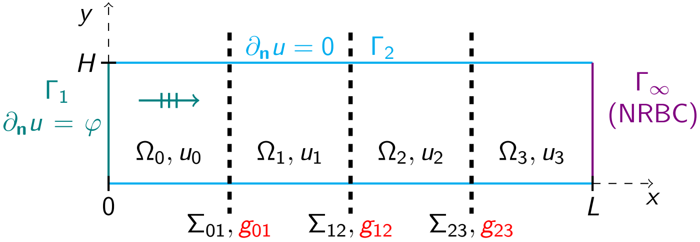

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<u>Schwarz substructured formulation</u> [*Gander et al. 2002*]

Iterative solver for interface problem $(\mathbb{I} - \Pi\mathbb{S})\mathbf{g} = \varphi$ on Σ

At each (n) iteration

1. Given $g_{ij}^{(n)}$, compute $u_i^{(n+1)}$ in Ω_i with direct solver $(\partial_{n_i} u_i + \imath\mathcal{S}_i u_i = g_{ij})$,
2. Update the interface unknowns on Σ_{ij}
   $g_{ji}^{(n+1)} = -g_{ij}^{(n)} + \imath\,(\mathcal{S}_i + \mathcal{S}_j)\, u_i^{(n+1)}$

If $(\mathcal{S}_i, \mathcal{S}_j) \approx$ outgoing DtN map $\Lambda^+ \rightarrow$ convergence in N_{dom} iterations

Convergence factor for convected propagation

Suppose a mean flow only along x-direction ($v_y = 0$)
We have **complex advection**: outgoing and incoming waves have a phase shift

$$\rho(\xi) = \left| \frac{(f - f_{n,\alpha})(-2M_x\omega + f - f_{n,\alpha})}{(-2M_x\omega + f + f_{n,\alpha})(f + f_{n,\alpha})} \right|, \; M_x = v_x/c_0$$

$f = \sqrt{1 + (1 - M_x^2)(\xi/\omega)^2}$, $f_{n,\alpha}$: square-root approx., ξ : Fourier variable

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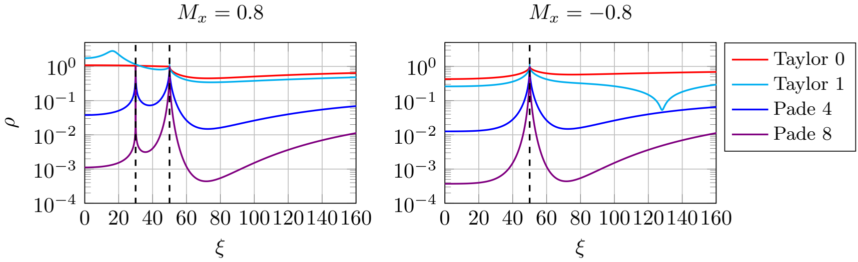


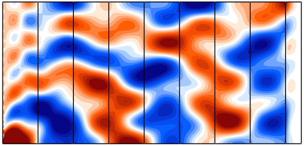
Convergence factor: $\alpha = -\pi/2, \omega = 30$. For $M_x = 0.8$, $\xi \in [30, 50]$, modes have negative phase velocity

How is numerical convergence affected ?

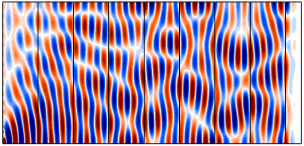
Assessment with ABC transmission operators

Absorbing boundary conditions as transmission operator, with $\alpha = -\pi/2$
The source is the superposition of the 30 first modes

Real part (+PML) $M_x = 0.8, \omega = 30$



Real part (+PML) $M_x = -0.8, \omega = 30$



$M_x = 0.8, \omega = 30$

N_{dom}	T0	T1	P8
2	20 (dnc)	18 (dnc)	3 (3)
4	60 (dnc)	58 (dnc)	9 (9)
8	142 (dnc)	133 (dnc)	19 (21)

$M_x = -0.8, \omega = 30$

N_{dom}	T0	T1	P8
2	14 (47)	10 (25)	3 (3)
4	44 (dnc)	28 (47)	7 (9)
8	94 (dnc)	62 (dnc)	13 (21)

Number of iterations to $r_l = 10^{-6}$: GMRES vs (Jacobi) solver. T: Taylor, P: Padé

- inverse upstream modes significantly deteriorates convergence
- Padé approximations reach high accuracy after N_{dom} iterations

Assessment with PML transmission operators

Let us use a PML for $(\mathcal{S}_i, \mathcal{S}_j)$, as approximations of Λ^+

Warning: "classic" PML is unstable for inverse upstream modes ($M_x > 0$)

$$
\begin{tikzcd}
\varphi(\mathbf{x},\omega) \arrow[r, "\text{Lorentz}"'] \arrow[d, dotted, "\text{unstable}"'] & \varphi'(\mathbf{x}',\omega') \arrow[d, "\text{PML}"] \\
\widetilde{\varphi}(\widetilde{\mathbf{x}},\omega) & \widetilde{\varphi}'(\widetilde{\mathbf{x}'},\omega') \arrow[l, "\text{Inverse Lorentz}"]
\end{tikzcd}
$$

[*Marchner et al. 2021*]

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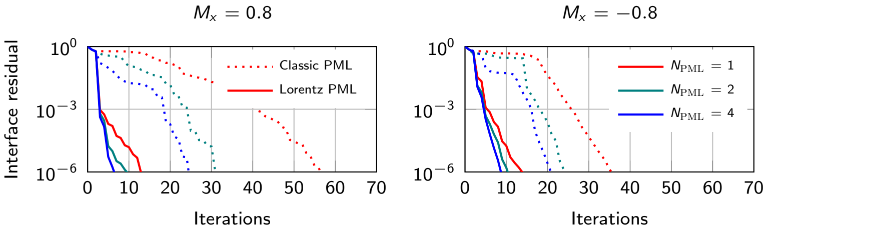
\end{tikzcd}$$

[*Marchner et al. 2021*]



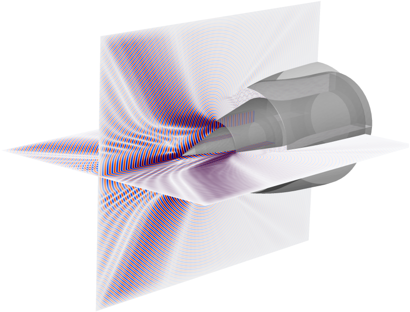
GMRES iterations, $N_{\mathrm{dom}} = 4, \omega = 40$

- With a Jacobi solver, a "classic" PML as transmission operator does not converge, even when $M_x < 0$!

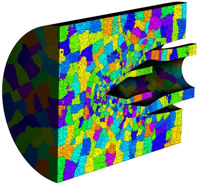
$\rightarrow$ Caution is needed with PML-transmission conditions [*Galkowski et al. 2024*]

Large scale Schwarz domain decomposition

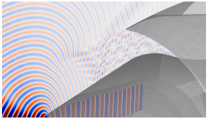
- For realistic problems we use 2nd order transmission condition
  Turbofan engine jet noise benchmark at $\omega/2\pi = 40$ kHz [*Marchner et al. 2025*]



(a) $\Re(u)$



(b) partitioning (METIS)



(c) Zoom

Run on Lumi on 65k cores: $N_{\text{dom}} = 4096$, 1.3×10^9 unknowns, 96×10^9 nnz
Peak memory over MPIs: 18.4 Gb, Its : 555 ($r_l < 10^{-4}$), solving time: 15min

Outline

Conclusion

I have presented domain decomposition and domain truncation techniques for convected propagation

- ABCs and PMLs can be extended to convected propagation, with high Mach numbers and convex boundary shape
- The Lorentz transform helps to better understand convected propagation
- Rational approx. of DtN maps are of strong interest for many PDEs
- Deriving corner conditions is currently an open problem

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All implementations were performed with `GmshDDM` and `GmshFEM` [*Royer et al. 2021*]

FEM DDM

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Thank you !

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FEM DDM