Towards an efficient domain decomposition solver for industrial time-harmonic flow acoustics

Philippe Marchner

Siemens Industry Software
philippe.marchner@siemens.com

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X. Antoine

C. Geuzaine

H. Bériot/P. Barabinot

Outline

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The linearized potential equation

Scalar equation for the acoustic velocity potential $u = \nabla \boldsymbol{v}$

Linearized Potential Equation (LPE)

$$\rho_0(\boldsymbol{x}) \frac{\mathrm{D}_0}{\mathrm{D}t} \left(\frac{1}{c_0(\boldsymbol{x})^2} \frac{\mathrm{D}_0 u}{\mathrm{D}t} \right) - \nabla \cdot (\rho_0(\boldsymbol{x}) \nabla u) = f, \quad \frac{\mathrm{D}_0}{\mathrm{D}t} = \mathrm{i}\omega + \boldsymbol{v}_0(\boldsymbol{x}) \cdot \nabla$$

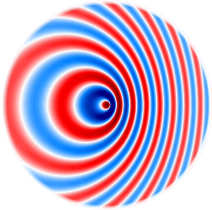
Helmholtz-type problem with convection and heterogeneities

Mathematical difficulties

- oscillatory, non-local solution
- complex valued, strongly indefinite with ω
- unbounded domain
- convection effects

hard to converge with classical iterative methods [*Ernst, Gander 2012*]

Point source in a uniform flow



$\mathrm{M} = \|\boldsymbol{v}_0\| / c_0 = 0.6$

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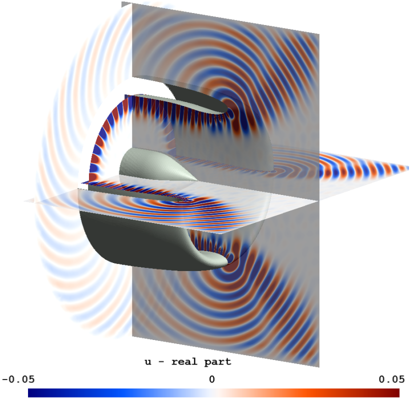
High frequency memory limit

Industrial example: single tone turbofan intake noise radiation
Current industrial solver: high-order p-FEM, direct solver for $\mathbb{K}\mathbf{u}_h = \mathbf{f}_h$

$\omega_{\text{bpf}} \leftrightarrow \approx$ 25 wavelengths



ω_{bpf}, $N_{\text{dofs}} = 10\text{M}$, nnz $= 730\text{M}$
Direct solver $\rightarrow$ 740 Gb of RAM

$\Downarrow$ increase ω ?

$2 \times \omega_{\text{bpf}}$, $N_{\text{dofs}} = 73\text{M}$, nnz $= 5\text{B}$
Direct solver $\approx$ 6 Tb of RAM ...

$\mathcal{O}(\omega^3)$ scaling in memory & time ...

can we distribute the memory cost ? $\rightarrow$ domain decomposition

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A parallel domain decomposition solver

For a domain partition $\Omega = \bigcup_{i=0}^{N_{\text{dom}}-1} \Omega_i$, solve in each subdomain Ω_i

Non-overlapping optimal Schwarz formulation

$$\begin{cases} \rho_0 \frac{D_0}{Dt}\left(\frac{1}{c_0^2}\frac{D_0 u_i}{Dt}\right) - \nabla\cdot(\rho_0 \nabla u_i) = 0 \text{ in } \Omega_i, \text{ (LPE)} \\ \rho_0\left(1-M_{\boldsymbol{n}}^2\right)\left(\partial_{\boldsymbol{n}_i} u_i + \imath\widetilde{\Lambda}^+ u_i\right) = 0, \text{ on } \Gamma_i^\infty \text{ (radiation condition)} \\ \rho_0\left(1-M_{\boldsymbol{n}}^2\right)\left(\partial_{\boldsymbol{n}_i} u_i + \imath\mathcal{S}_i u_i\right) = g_{ij}, \text{ on } \Sigma_{ij}, \text{ (interface condition)} \end{cases}$$

Introduce the interface coupling on Σ_{ij}

$$\begin{aligned} g_{ij} &= \rho_0\left(1-M_{\boldsymbol{n}}^2\right)\left(-\partial_{\boldsymbol{n}_j} u_j + \imath\mathcal{S}_i u_j\right) \\ &= -g_{ji} + \imath\rho_0\left(1-M_{\boldsymbol{n}}^2\right)(\mathcal{S}_i + \mathcal{S}_j)u_j := \mathcal{T}_{ji} g_{ji} + b_{ji} \end{aligned}$$

Rewrite the coupling as a linear system for $\boldsymbol{g} = (g_{ij}, g_{ji})^T$

$$\underbrace{(\mathcal{I}-\mathcal{A})}_{\text{iteration matrix}} \underbrace{\boldsymbol{g}}_{\text{interface unknowns}} = \underbrace{\boldsymbol{b}}_{\text{physical sources}}, \quad \mathcal{A} = \begin{pmatrix} 0 & \mathcal{T}_{ji} \\ \mathcal{T}_{ij} & 0 \end{pmatrix}$$

$\mathcal{T}_{ij}$ and $\mathcal{T}_{ji}$ are the iteration operators, and can be written in terms of $\widetilde{\Lambda}^+$

High-level algorithmic procedure

Surfacic iteration operators

$$\mathcal{T}_{ji} = \frac{\mathcal{S}_i - \widetilde{\Lambda}^+}{\mathcal{S}_j + \widetilde{\Lambda}^+}, \quad \mathcal{T}_{ij} = \frac{\mathcal{S}_j + \widetilde{\Lambda}^-}{\mathcal{S}_i - \widetilde{\Lambda}^-}, \quad \text{on } \Sigma_{ij}$$

Iteration matrix eigenvalues: $\lambda_{(\mathcal{I}-\mathcal{A})} = 1 \pm \sqrt{\mathcal{T}_{ji}\mathcal{T}_{ij}}$

If we choose $\mathcal{S}_i = \widetilde{\Lambda}^+$ and $\mathcal{S}_j = -\widetilde{\Lambda}^-$, we converge in N_{dom} iterations

Parallel iterative algorithm for the process i

Do in Ω_i at iteration $(n+1)$, $\forall j \in D_i$

1. given $g_{ij}^{(n)}$, solve $u_i^{(n+1)}$ in Ω_i,
2. update the $(n+1)$ neighbourhood data through $g_{ji}^{(n+1)} = -g_{ij}^{(n)} + \imath\rho_0 \left(1 - M_{\boldsymbol{n}}^2\right) (\mathcal{S}_i + \mathcal{S}_j) u_i^{(n+1)}$ on Σ_{ij},

High-level algorithmic procedure

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$(\widetilde{\Lambda}^+, -\widetilde{\Lambda}^-)$ are **non-local** DtN maps for the LPE

$\rightarrow$ design **sparse approximations** $\mathcal{S}_i \approx \widetilde{\Lambda}^+$ and $\mathcal{S}_j \approx -\widetilde{\Lambda}^-$

$\Leftrightarrow$ approximate Schur complements at the algebraic level

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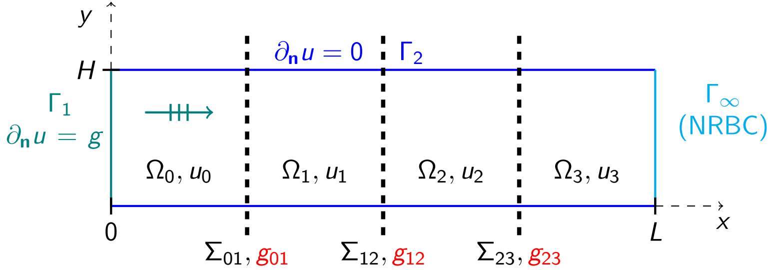
Waveguide problem with straight partitions

Domain partition: $\Omega = \bigcup_{i=0}^{N_{\text{dom}}-1} \Omega_i$, $\Sigma_{ij} = \overline{\partial\Omega_i \bigcap \partial\Omega_j}$, $j \neq i$



Example: transverse heterogeneous Helmholtz problem $c_0(y), \rho_0(y)$

$$\partial_x^2 u + \rho_0(y)^{-1} \partial_y \left(\rho_0(y) \partial_y\right) u + \omega^2 c_0(y)^{-2} u = 0$$

Half-space DtN map for positive outgoing waves

$$\widetilde{\Lambda^+} = +\sqrt{\omega^2 c_0^{-2} + \rho_0^{-1} \nabla_\Gamma \left(\rho_0 \nabla_\Gamma\right)}$$

$\widetilde{\Lambda^+}$ is pseudo-differential, we need local approximations

Local square-root approximation

Two points of view: $\omega \to +\infty$ or $k_0 \to +\infty$, $k_0 = \omega / c_0(y)$

Operator choice before localization *[Marchner et al. SIAP 2022]*

$$\Lambda_\omega = \omega \sqrt{1 + \left[\left(c_0^{-2} - 1 \right) + \frac{\rho_0^{-1} \nabla_\Gamma \left(\rho_0 \nabla_\Gamma \right)}{\omega^2} \right]}, \quad \Lambda_{k_0} = k_0 \sqrt{1 + \frac{\Delta_\Gamma}{k_0^2}}$$

Use complexified Padé or Taylor approximations (N, α) to localize

- order of approximation N,
- square-root rotation branch-cut of angle α *[Milinazzo et al. 1997]*
  $\to$ captures both propagative *and* evanescent modes

$$\Lambda(z) = e^{\mathrm{i}\alpha/2} \sqrt{1 + z}, \; z = [e^{-\mathrm{i}\alpha}(1 + X) - 1]$$

A family of local DDM transmission conditions $\mathcal{S}_i$

$\Rightarrow$ Padé-based: $\mathrm{ABC}_\omega^{N,\alpha}$ and $\mathrm{ABC}_{k_0}^{N,\alpha}$

$\Rightarrow$ Taylor-based: $\mathrm{ABC}^{\mathrm{T0},\alpha}$, $\mathrm{ABC}_{k_0}^{\mathrm{T2},\alpha}$, $\mathrm{ABC}_\omega^{\mathrm{T2},\alpha}$

Gaussian waveguide - Iteration matrix

Gaussian profile $c_0(y) = 1.25\left(1 - 0.4e^{-32(y-H/2)^2}\right), \ \rho_0(y) = c_0^2(y)$

Iteration matrix eigenvalues: $\lambda_{(\mathcal{I}-\mathcal{A})} = 1 \pm \sqrt{\mathcal{T}_{ji}(y)\mathcal{T}_{ij}(y)}$

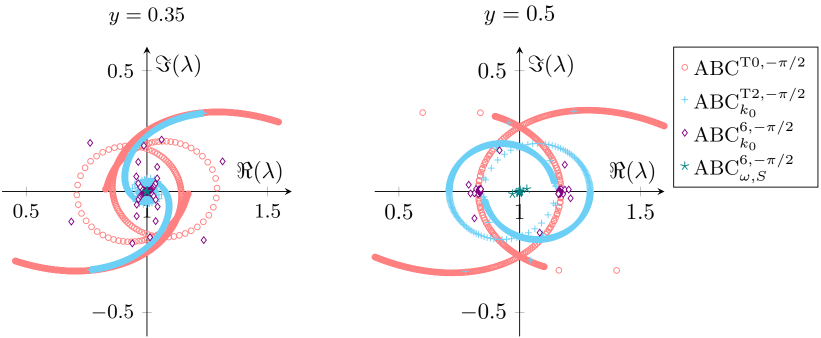
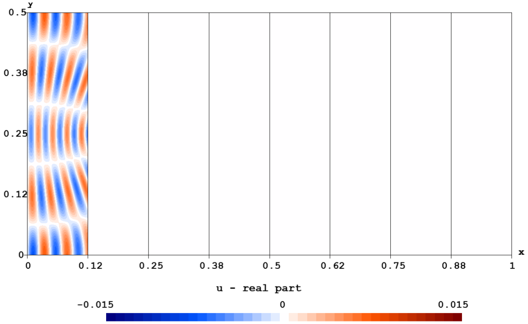


Figure: Theoretical eigenvalues of the DDM iteration matrix, $\omega = 50$, $H = 1$

- Usual Padé approximation - $\mathcal{S}_i = \mathrm{ABC}_{k_0}^{N,\alpha}$
- New Padé approximation - $\mathcal{S}_i = \mathrm{ABC}_{\omega,S}^{N,\alpha}$ → almost perfect clustering

Large PML on Γ_∞ - input mode $n = 4$ - $N_{\text{dom}} = 8$



N_{dom}	$\text{ABC}_{k_0}^{\text{T0},-\pi/4}$	$\text{ABC}_{k_0}^{\text{T2},-\pi/4}$	$\text{ABC}_{k_0}^{4(8),-\pi/4}$	$\text{ABC}_{\omega}^{4(8),-\pi/4}$
8	111	74	42 (42)	20 (8)

Table: GMRES iterations to 10^{-6} at $\omega = 160$, $d_\lambda = 24$

Convergence in N_{dom} iterations $\rightarrow$ continuous block LU factorization

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DtN map for the Linearized Potential Equation

Wave convection by a steady subsonic mean flow $M < 1$

$$\mathcal{L}(\boldsymbol{x}, \nabla, \omega) = \frac{D_0}{Dt}\left(\frac{1}{c_0^2}\frac{D_0}{Dt}\right) - \rho_0^{-1}\nabla \cdot (\rho_0 \nabla), \quad \frac{D_0}{Dt} = i\omega + \mathbf{v}_0 \cdot \nabla$$

DtN principal symbol (half-space), $M_x = v_{0,x}/c_0$, $M_y = v_{0,y}/c_0$, $k_0 = \omega/c_0$

$$\lambda_1^+ = \frac{1}{1 - {M_x}^2}\left[-M_x(k_0 - \xi M_y) + \sqrt{k_0^2 - 2k_0 M_y \xi - (1 - M^2)\xi^2}\right]$$

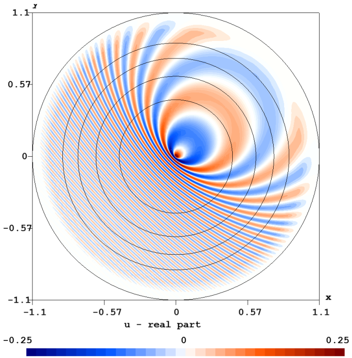
valid in the tangent plane approximation, $M_n = \mathbf{v}_0 \cdot \mathbf{n}/c_0$, $M_\tau = \mathbf{v}_0 \cdot \boldsymbol{\tau}/c_0$

$$\widetilde{\Lambda}^+ \approx \widetilde{\Lambda}_1^+ = \mathrm{Op}(\lambda_1^+) = \frac{k_0}{1 - {M_n}^2}\left(-M_n + iM_n M_\tau \frac{\nabla_\Gamma}{k_0} + \sqrt{1 + X}\right),$$

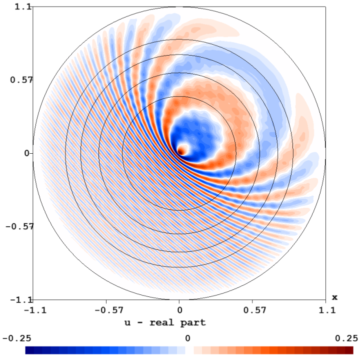
$$X = -2iM_\tau \frac{\nabla_\Gamma}{k_0} + (1 - M^2)\frac{\Delta_\Gamma}{k_0^2}, \quad M = \|\mathbf{v}_0\|/c_0$$

Complex Padé approximants for $\sqrt{1+X} \Rightarrow \mathrm{ABC}_1^{N,\alpha}$
Complex Taylor approximants $\Rightarrow \mathrm{ABC}_1^{\mathrm{T0},\alpha}$ and $\mathrm{ABC}_1^{\mathrm{T2},\alpha}$

Convected problem in freefield - circular interfaces



(a) $\text{ABC}_1^{4,-\pi/4}$, $\mathcal{E}_{L^2} = 1.7\%$



(b) $\text{ABC}_1^{\text{T}2,-\pi/4}$, $\mathcal{E}_{L^2} = 24\%$

Numerical solution after 4 GMRES DDM iterations.
Parameters: $M = 0.9$, FEM-order $= 9$, $d_\lambda = 8$, $N_{\text{dom}} = 5$, $\omega = 6\pi$.

Padé conditions are robust for **high Mach numbers**

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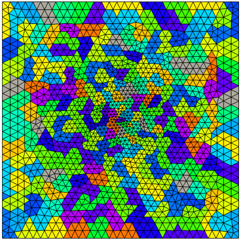
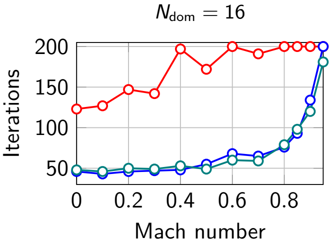
Convected problem in freefield - arbitrary decomposition

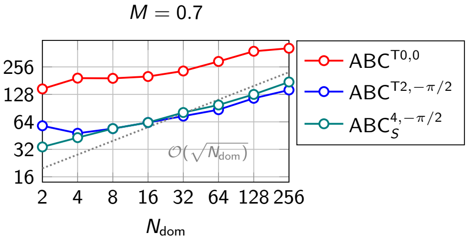
Automatic partitioning

- Cross-points
  $\rightarrow$ harder to design ABCs
- Industrial need - good load balancing between subproblems
- Shorter connectivity graph - $\mathcal{O}(\sqrt{N_{\text{dom}}})$

$N_{\text{dom}} = 256$







We choose $\text{ABC}^{\text{T}2,-\pi/2}$ for arbitrary decomposition

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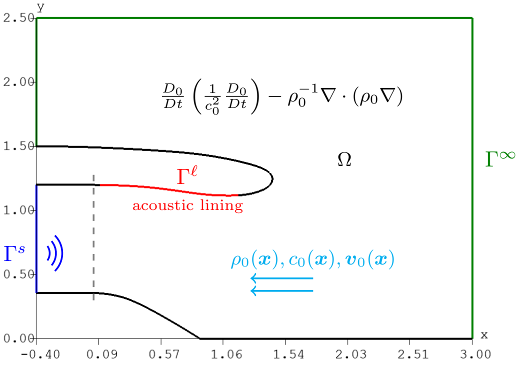
Given a flight configuration (mean flow), predict the radiated noise from the fan, at multiples of the blade passing frequency $\omega_{\text{bpf}}/(2\pi) = 1300$ Hz



Boundary conditions

- Ingard-Myers on Γ_ℓ
- PML (active) on Γ^s
- Fixed annular Bessel mode on Γ^s
- PML (passive) on Γ^∞

The mean flow is pre-computed and **interpolated** on the acoustic mesh

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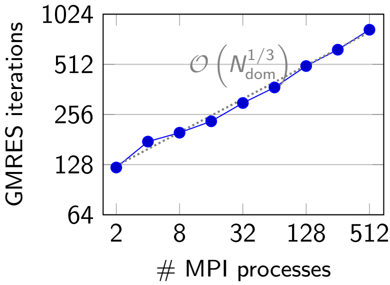
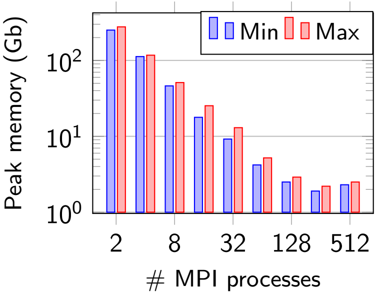
DDM for the 3D turbofan problem

ω_{bpf} - $N_{\text{dofs}} = 10\text{M}$ - nnz $= 730\text{M}$ $\approx$ 25 wavelengths in Ω

Direct solver $\rightarrow$ 740 Gb RAM for factorization





Parallel GmshDDM solver (mono-thread)

From $N_{\text{dom}} > 128$, under 10 minutes and less than 3Gb per process

Iteration number for $N_{\text{dom}} = 64$

- $\text{ABC}^{\text{T}2,-\pi/2}$: 372 GMRES iterations to reach 10^{-6}
- $\text{ABC}^{\text{T}0,0}$: $>$ 2000 GMRES iterations to reach 10^{-3}

DDM for the 3D turbofan problem

$2 \times \omega_{\text{bpf}}$ - $N_{\text{dofs}} = 73\text{M}$ - nnz $= 5\text{B}$ $\approx$ 50 wavelengths in Ω

Parallel GmshDDM solver (mono-thread), $N_{\text{dom}} = \#$ MPI $= 128$

$\rightarrow$ 2hours with 26 Gb peak memory, 712 GMRES iterations (with lining)

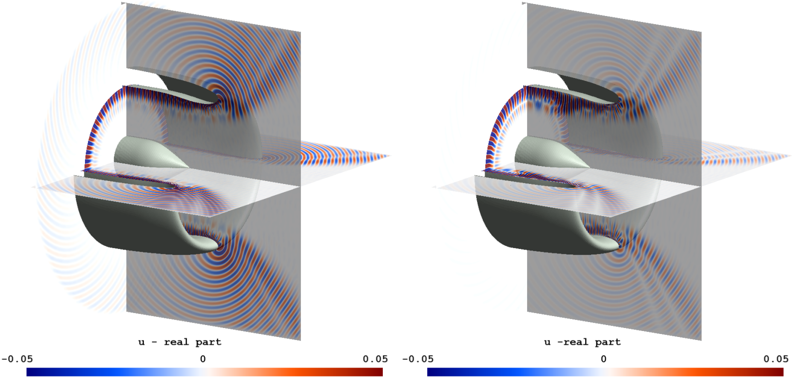


Figure: Real part of the acoustic velocity potential for the mode $(48, 1)$ at $2 \times \omega_{\text{bpf}}$ (2600 Hz) without (left) and with (right) acoustic lining treatment.

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Solver weak scalability

3D Helmholtz problem with $d_\lambda = 7.5$ points per wavelength

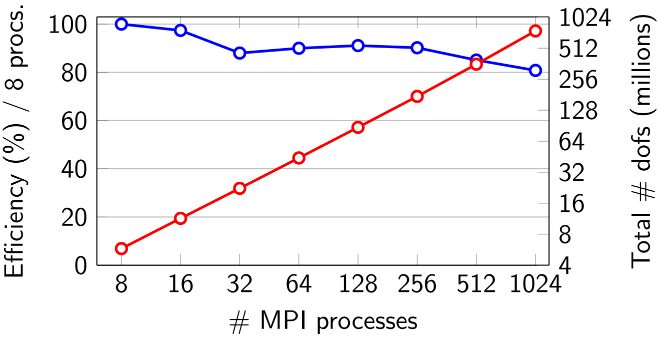

Figure: Weak scaling timing for 1 iteration (LUMI CPU partition)

Limitations

- memory load balancing: [20-34] Gb on 1024 processes
- number of iterations scales as $\mathcal{O}(N_{\text{dom}}^{1/3})$ in 3D

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Conclusion

Summary

- Non-overlapping Schwarz domain decomposition for flow acoustics
  → distributed memory solver
- Transmission conditions with convection and heterogeneities
- Proof of concept for a 3D turbofan industrial problem
  → further computations are ongoing

Limitations

- The iterations does not scale with N_{dom} (coarse space is needed)
- Cross-points, broken-curved boundaries hamper the convergence

Perspectives

- Modern volumic discretization techniques: HDG, HHO, adaptive p-FEM [*Bériot et al. 2016, 2019*]
- Extension to Pierce equation → turbofan exhaust [*Spieser, Bailly 2020*]