Local absorbing boundary conditions for heterogeneous and convected time-harmonic acoustic problems

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Outline

Motivation

Microlocal analysis

Absorbing boundary conditions on academic examples
- Longitudinal heterogeneous waveguide
- Transverse heterogeneous waveguide
- Convected Helmholtz operator

Application to non-overlapping domain decomposition

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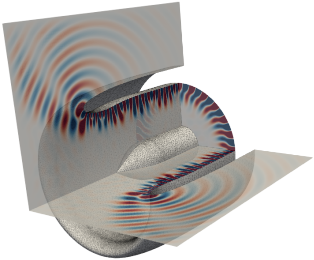
Motivation

Non-reflecting boundary conditions for time-harmonic problems
$\rightarrow$ Well-developed in the Helmholtz case (ABCs, PMLs, Infinite elements, etc.)

Industrial situation - flow acoustics

- heterogeneous medium
  spatially varying density $\rho_0(\boldsymbol{x})$ and speed of sound $c_0(\boldsymbol{x})$
- convection
  spatially varying velocity vector field $\mathbf{v}_0(\boldsymbol{x})$



Goals

1. improve accuracy of ABCs for such problems [*Marchner et al., to appear in SIAP, 2022*],
2. use them as transmission conditions for non-overlapping Schwarz domain decomposition
   $\rightarrow$ parallel time-harmonic solver [*Lieu et al., CMAME, 2020*]

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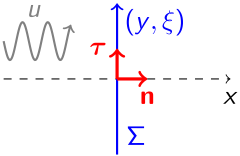
Microlocal analysis - DtN operator

ABCs: find local approximations of the **Dirichlet-to-Neumann map**

$$\widetilde{\Lambda^+} : \begin{cases} H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma) \\ u_{|\Sigma} \mapsto \widetilde{\Lambda^+} u_{|\Sigma} = \partial_{\mathbf{n}} u_{|\Sigma} \end{cases},$$



through pseudo-differential calculus [*Engquist and Majda 1977 & 1979*]

2D Helmholtz half-space problem with heterogeneous coefficients

$$\mathcal{L} = \rho_0^{-1}\partial_x(\rho_0\partial_x) + \rho_0^{-1}\partial_y(\rho_0\partial_y) + \omega^2 c_0^{-2}$$
$$\underset{?}{\approx} \left(\partial_x - i\sqrt{\omega^2 c_0^{-2} + \rho_0^{-1}\partial_y(\rho_0\partial_y)}\right)\left(\partial_x + i\sqrt{\omega^2 c_0^{-2} + \rho_0^{-1}\partial_y(\rho_0\partial_y)}\right)$$

We cannot formally factorize $\mathcal{L}$ when $\partial_x(\rho_0) \neq 0$ or $\partial_x(c_0) \neq 0$

- Compute instead the **symbol** λ^+ of Λ^+

$\rightarrow$ work on polynomials in ξ with pseudo-differential algebraic rules

Microlocal analysis - symbol expansion

Nirenberg's factorization theorem: there exists $(\Lambda^+, \Lambda^-) \in \mathrm{OPS}^1$

$$\mathcal{L} = \left(\partial_x + i\Lambda^+\right)\left(\partial_x + i\Lambda^-\right) \quad \text{mod } \mathrm{OPS}^{-\infty}$$
$$= \partial_x^2 + i\left(\Lambda^+ + \Lambda^-\right)\partial_x + i\mathrm{Op}\left\{\partial_x \lambda^+\right\} - \Lambda^-\Lambda^+ \quad \text{mod } \mathrm{OPS}^{-\infty}.$$

Identify with the Helmholtz operator and get an equation for Λ^+

$$\left(\Lambda^+\right)^2 + i\rho_0^{-1}\partial_x(\rho_0)\Lambda^+ + i\mathrm{Op}\left\{\partial_x\lambda^+\right\} = \omega^2 c_0^{-2} + \rho_0^{-1}\partial_y(\rho_0\partial_y)$$

"high frequency" asymptotic expansion for the total symbol λ^+

$$\lambda^+ \sim \sum_{j=-1}^{\infty} \lambda^+_{-j} = \lambda^+_1 + \lambda^+_0 + \lambda^+_{-1} + \cdots \text{ (classical symbol expansion)}$$

Each λ^+_{-j} is homogeneous of leading order $(\omega/c_0, \xi)^{-j}$

$$\lambda_1^+ = \sqrt{\omega^2 c_0^{-2} - \xi^2},\ \lambda_0^+ = -i\left(\frac{\partial_x(\rho_0)}{2\rho_0} + \frac{\xi\partial_y(\rho_0)}{2\rho_0\lambda_1^+} + \frac{\omega^2\partial_x\left(c_0^{-2}\right)}{4\left(\lambda_1^+\right)^2} + \frac{\xi\omega^2\partial_y\left(c_0^{-2}\right)}{4\left(\lambda_1^+\right)^3}\right)$$

Microlocal regimes

Outgoing propagating waves are characterized by $\Re(\lambda_1^+) > 0$

Evanescent waves ? rotate the square root branch-cut [*Milinazzo et al. 1997*]

$$\lambda_1^+ = e^{i\alpha/2}\sqrt{e^{-i\alpha}(\omega^2 c_0^{-2} - \xi^2)},\ \alpha \in [0, -\pi],\ (+i\omega t \text{ convention})$$

- $\omega c_0^{-1} > |\xi|$, x-positive propagating wave $\rightarrow \alpha = 0$,
- $\omega c_0^{-1} < |\xi|$, x-positive evanescent wave $\rightarrow \alpha = -\pi$,
- $\omega c_0^{-1} = |\xi|$, grazing wave, ill-posed problem

In practice choose $\alpha \in [0, -\pi/2]$

$\rightarrow$ trade-off to capture both propagative and evanescent part of the spectrum

Summary - building the DtN approximation

Approximate DtN operator $\widetilde{\Lambda}_M^+$

$$\partial_{\mathbf{n}} u = -i\Lambda_M^+ u, \quad \Lambda_M^+ = \sum_{j=-1}^{M-2} \mathrm{Op}\left(\lambda_{-j}^+\right)$$

Take the trace on the outgoing boundary Σ to get $\widetilde{\Lambda}_M^+$

Difficulties

- the formal computation of λ_{-j}^+ can be involved (PDE dependent),
- the operator $\mathrm{Op}\left(\lambda_{-j}^+\right)$ is in general not unique and still non-local
- limited *a priori* to smooth variations of ρ_0 and c_0

Next step

- focus on simplified academic situations

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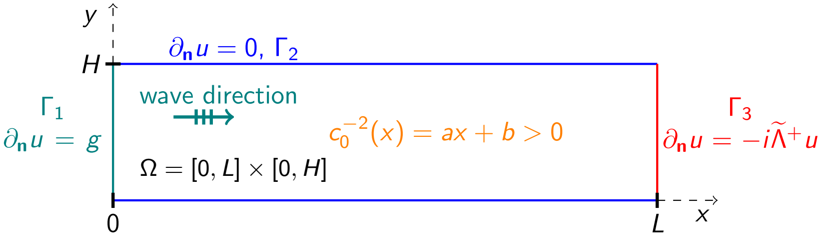
Longitudinal heterogeneous waveguide

Single mode propagation in a heterogeneous waveguide: $\rho_0 = 1$, $c_0(x, y) = c_0(x)$



Single mode analytic solution for a linear profile

$$u^n_{\mathrm{ex}}(x, y) = \cos\left(k_y y\right) \mathrm{Ai}\left(e^{-\frac{2i\pi}{3}} \frac{k_y^2 - \omega^2(ax+b)}{(a\omega^2)^{2/3}}\right), \quad k_y = \frac{n\pi}{H}, \quad n \in \mathbb{N}$$

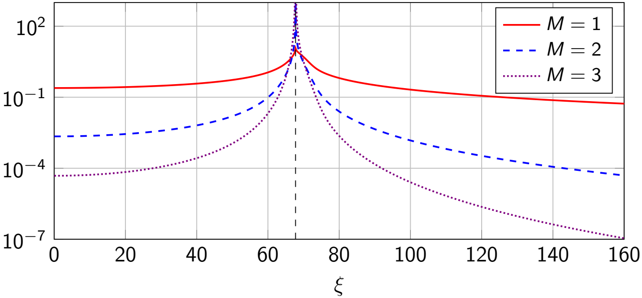
Approximation at the symbol level

$c_0^{-2}(x) = 5x + 0.1$, $L = 1$ at fixed frequency $\omega = 30$

Analytic total symbol

$$\lambda^+ = -ie^{-\frac{2i\pi}{3}} \left(a\omega^2\right)^{1/3} \frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}, \quad z = e^{-\frac{2i\pi}{3}} \frac{\xi^2 - \omega^2(aL+b)}{(a\omega^2)^{2/3}}$$

$$\left|\lambda^+ - \sum_{j=-1}^{M-2} \lambda^+_{-j}\right|$$



- singularity in the grazing regime $\xi \approx \omega c_0^{-1}$

Operator representation

Truncate the asymptotic expansion for λ^+

$$\lambda^+ \approx \lambda_1^+ + \lambda_0^+ = \sqrt{\omega^2 c_0^{-2} - \xi^2} - i \frac{\omega^2 \partial_x \left(c_0^{-2}\right)}{4\left(\omega^2 c_0^{-2} - \xi^2\right)}$$

Unique operator representation on $\Gamma_3 := \{x = L,\ y \in [0, H]\}$

$$\widetilde{\Lambda}_1^+ = \mathrm{Op}(\lambda_1^+) = \sqrt{\omega^2 c_0^{-2} + \Delta_\Gamma}$$

$$\widetilde{\Lambda}_2^+ = \mathrm{Op}(\lambda_1^+ + \lambda_0^+) = \sqrt{\omega^2 c_0^{-2} + \Delta_\Gamma} - i \frac{\omega^2 \partial_x \left(c_0^{-2}\right)}{4} \left(\omega^2 c_0^{-2} + \Delta_\Gamma\right)^{-1}$$

$\widetilde{\Lambda}_1^+$ and $\widetilde{\Lambda}_2^+$ are still **non-local** $\Rightarrow$ square-root localization

Diagonal Padé approximants of order N in the high frequency limit

$$\sqrt{1+X} \approx K_0 + \sum_{\ell=1}^{N} A_\ell X (1 + B_\ell X)^{-1},\ X = \Delta_\Gamma / (\omega^2 c_0^{-2}) \to 0$$

High-order FEM implementation

Weak formulation: discretization on a conformal high-order H^1-basis

$$\forall v \in H^1(\Omega), \quad \int_\Omega \left\{ \nabla u \cdot \nabla \bar{v} - \omega^2 c_0^{-2} u \bar{v} \right\} d\Omega + i \int_{\Gamma_3} \widetilde{\Lambda}_2^+ u \bar{v} d\Gamma = \int_{\Gamma_1} g \bar{v} d\Gamma$$

$$\int_{\Gamma_3} \widetilde{\Lambda}_2^+ u \overline{v}\, d\Gamma = \int_{\Gamma_3} \omega c_0^{-1} \underbrace{\sqrt{1+X}}_{\approx K_0 + \sum_{\ell=1}^N A_\ell X (1+B_\ell X)^{-1}} u \overline{v}\, d\Gamma - i \int_{\Gamma_3} \frac{\partial_x \left(c_0^{-2}\right)}{4 c_0^{-2}} (1+X)^{-1} u \overline{v}\, d\Gamma$$

Introduce **auxiliary fields** $\varphi_\ell = (1 + B_\ell X)^{-1} u$ and $\psi = (1+X)^{-1} u$

Sparse discretization of inverse operators
$\Rightarrow$ Augmented system of $N+1$ surfacic equations

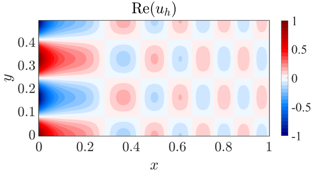
$$\int_{\Gamma_3} (1 + B_\ell X) \varphi_\ell \overline{v}_\ell \, d\Gamma = \int_{\Gamma_3} u \overline{v}_\ell \, d\Gamma$$

$$\int_{\Gamma_3} (1 + X) \psi \overline{\mu} \, d\Gamma = \int_{\Gamma_3} u \overline{\mu} \, d\Gamma$$

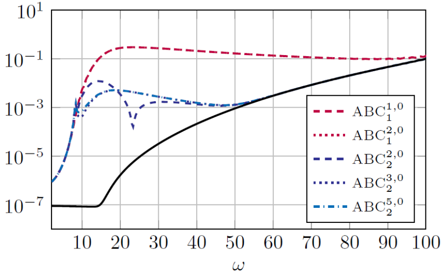
Numerical results

$\text{ABC}_M^{N,\alpha}$: Local Padé approximation of $\widetilde{\Lambda}_M^+$ with rotation branch-cut α

$\omega = 20$, $x_t = 0.16$, $n = 3$



Relative L^2-error (%)



use the derivative of c_0 ⇒ more accurate ABC: $\text{ABC}_2^{N+1,\alpha} > \text{ABC}_1^{N,\alpha}$

Gain of ≈ two order of magnitude
Precision limited by the DtN symbol truncation

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Transverse heterogeneous waveguide

Spatial variation transverse to the boundary: $c_0(y)$, $\rho_0(y)$

$$\partial_x^2 u + \rho_0^{-1}\partial_y\left(\rho_0\partial_y\right)u + \omega^2 c_0^{-2} u = 0$$

Formal operator factorization for outgoing waves

$$\Lambda_S = \sqrt{\omega^2 c_0^{-2} + \rho_0^{-1}\nabla_\Gamma\left(\rho_0\nabla_\Gamma\right)}$$

Microlocal justification - total symbol of Λ_S

$$\lambda_S^2 = \omega^2 c_0^{-2} - \xi^2 - i\rho_0^{-1}\partial_y(\rho_0)\xi$$

Classical symbol asymptotic expansion

$$\lambda_{1,S} = \sqrt{\omega^2 c_0^{-2} - \xi^2},\ \lambda_{0,S} = -i\xi\left(\frac{\partial_y\left(\rho_0\right)}{2\rho_0\lambda_{1,S}} + \frac{\omega^2\partial_y\left(c_0^{-2}\right)}{4\left(\lambda_{1,S}\right)^3}\right),\ \lambda_{-1,S} = \cdots$$

It exactly matches the total DtN symbol λ^+

$$\mathrm{Op}\left(\lambda^+\right) = \Lambda_S \bmod \mathrm{OPS}^{-\infty}$$

Square-root approximations

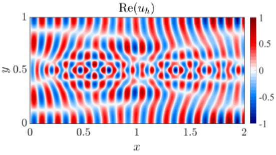
Example: Gaussian speed of sound profile: $c_0(y), \rho_0 = 1, \quad k_0 = \omega c_0^{-1}$

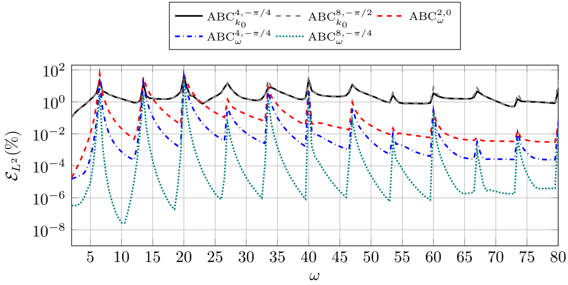
Operator choices for local square-root approximation

$$\Lambda_\omega = \omega\sqrt{1 + \left[(c_0^{-2} - 1) + \frac{\Delta_\Gamma}{\omega^2}\right]}, \quad \Lambda_{k_0} = k_0\sqrt{1 + \frac{\Delta_\Gamma}{k_0^2}}$$

Complexified Padé approximants for $\sqrt{1+X} \Rightarrow \mathrm{ABC}_\omega^{N,\alpha}$ and $\mathrm{ABC}_{k_0}^{N,\alpha}$

$\omega = 55, \ n = 4$



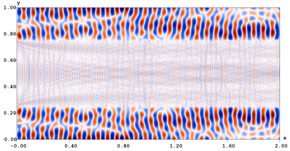


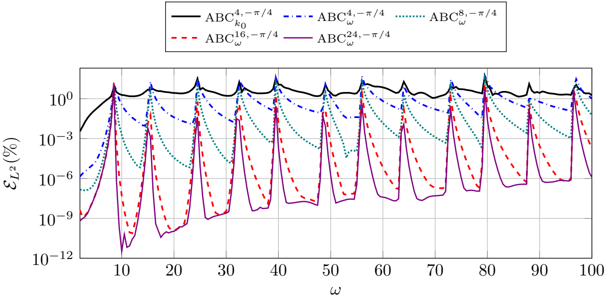
$\mathrm{ABC}_\omega^{N,\alpha}$ seems increasingly accurate except for grazing modes

Discontinuous speed of sound profile

Layered waveguide: $c_0(y) = \begin{cases} 1/4, & y \in [H/2-\delta, H/2+\delta] \\ 1, & \text{elsewhere} \end{cases}$, $\delta = H/4$

$\omega = 90,\ n = 4$





Limitations

- grazing modes
- can require a large number of auxiliary fields

Outline

Convected Helmholtz operator

Wave convection by a steady subsonic mean flow $M < 1$

$$\mathcal{L}(x, \partial_x, \omega) = \frac{D_0}{Dt}\left(\frac{1}{c_0^2}\frac{D_0}{Dt}\right) - \rho_0^{-1}\nabla \cdot (\rho_0 \nabla), \quad \frac{D_0}{Dt} = i\omega + \mathbf{v}_0 \cdot \nabla$$

Principal symbol (half-space), $M_x = v_{0,x}/c_0,\ M_y = v_{0,y}/c_0,\ k_0 = \omega/c_0$

$$\lambda_1^+ = \frac{1}{1 - {M_x}^2}\left[-M_x(k_0 - \xi M_y) + \sqrt{k_0^2 - 2k_0 M_y \xi - (1 - M^2)\xi^2}\right]$$

Tangent plane approximation, $M_n = \mathbf{v}_0 \cdot \mathbf{n}/c_0,\ M_\tau = \mathbf{v}_0 \cdot \boldsymbol{\tau}/c_0$

$$\widetilde{\Lambda}_1^+ = \mathrm{Op}(\lambda_1^+) = \frac{k_0}{1 - {M_n}^2}\left(-M_n + iM_n M_\tau \frac{\nabla_\Gamma}{k_0} + \sqrt{1 + X}\right),$$

$$X = -2iM_\tau \frac{\nabla_\Gamma}{k_0} + \left(1 - M^2\right)\frac{\Delta_\Gamma}{k_0^2}, \quad M = \|\mathbf{v}_0\|/c_0$$

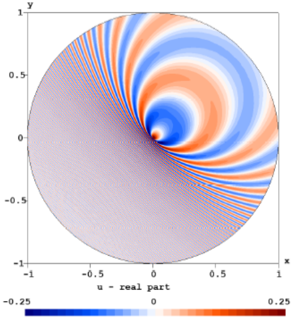
Complex Padé approximants of $\sqrt{1+X}$ for $\widetilde{\Lambda}_1^+ \Rightarrow \mathrm{ABC}_1^{N,\alpha}$

Complex Taylor approximants $\Rightarrow \mathrm{ABC}_1^{T0,\alpha}$ and $\mathrm{ABC}_1^{T2,\alpha}$

Attempt to incorporate curvature effects from λ_0^+ (circle of radius R)

$$\mathrm{ABC}_2 = \mathrm{ABC}_1 + (1-M^2)/(2R)$$

$k_0 = 6\pi$, $M = 0.95$, $R = 1$



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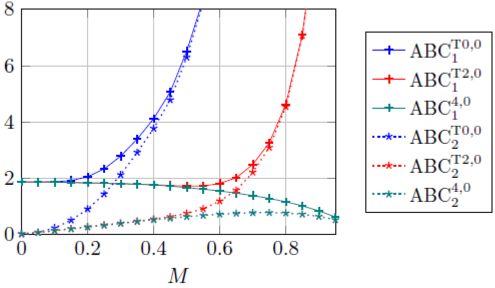
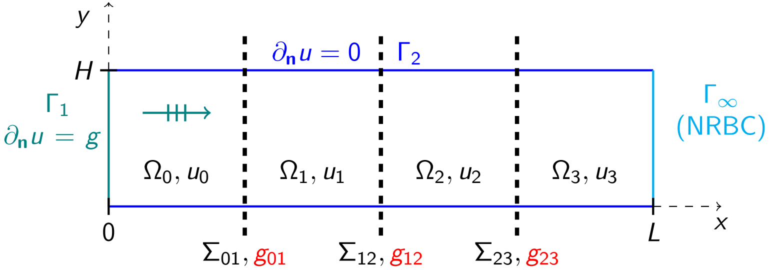


$\mathrm{ABC}_1^{N,\alpha}$ is robust for high Mach numbers - wavelength ratio $(1+M)/(1-M)$

Outline

Domain decomposition in a nutshell

Domain partition: $\Omega = \bigcup_{i=0}^{N_{\text{dom}}-1} \Omega_i$, $\Sigma_{ij} = \overline{\partial\Omega_i \bigcap \partial\Omega_j}$, $j \neq i$



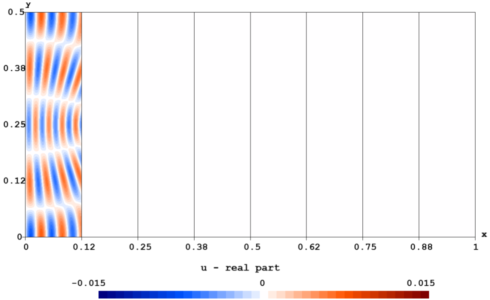
Parallel iterative (e.g. GMRES) solver for $\mathcal{A}\mathbf{g} = \mathbf{f}$ on Σ $(\partial_{n_i} u_i + i\mathcal{S}_i u_i = g_{ij})$
Do at iteration (n)

1. Given $g_{ij}^{(n)}$, solve by a direct method $u_i^{(n)}$ in Ω_i,
2. Update the $(n+1)$ interface unknowns on Σ_{ij} thanks to
   $g_{ji}^{(n+1)} = -g_{ij}^{(n)} + i(\mathcal{S}_i + \mathcal{S}_j)u_i^{(n+1)}$

If we choose $\mathcal{S}_i = \widetilde{\Lambda}^+$ and $\mathcal{S}_j = \widetilde{\Lambda}^-$, we converge in $(N_{\text{dom}} - 1)$ iterations

ABCs for $\mathcal{S}_i$ and $\mathcal{S}_j$: propagating, evanescent and grazing modes

Gaussian waveguide: $c_0(y) = 1.25\left(1 - 0.4e^{-32(y-H/2)^2}\right), \rho_0(y) = c_0^2(y)$



$\mathcal{S}_i$	$\text{ABC}_{k_0}^{\text{T0},-\pi/4}$	$\text{ABC}_{k_0}^{\text{T2},-\pi/4}$	$\text{ABC}_{k_0}^{4(8),-\pi/4}$	$\text{ABC}_{\omega}^{4(8),-\pi/4}$
It $(r_I = 10^{-6})$	110	73	41 (41)	19 (7)

Conclusion

- ABCs
  - microlocal based ABCs for some heterogeneous and convected Helmholtz problems
  - the extension to more complex situations is challenging: corners, combine different variations, curved boundaries, etc.
- Non-overlapping DDM
  - improved DDM convergence with quasi-local operators for simple shaped interfaces

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Thank you ! Questions ?